

Quantum Field Theory in the Language of Light-cone String¹

Charles B. Thorn²

*Institute for Fundamental Theory, Department of Physics
University of Florida, Gainesville FL 32611*

Abstract

The worldsheet representation of the sum of the planar diagrams of scalar Φ^3 field theory and $\mathcal{N} = 0, 1, 2, 4$ supersymmetric Yang-Mills theory is explained. This was a talk given to the Light Cone Workshop: Hadrons and Beyond, 5-9 August 2003, University of Durham.

1 Introduction

One of the most striking dualities that has emerged in the development of string theory is the so-called AdS/CFT correspondence [1, 2] in which the $N_c = \infty$ limit of the conformal invariant $\mathcal{N} = 4$ extended supersymmetric Yang-Mills theory is conjectured to be equivalent to a noninteracting IIB superstring theory on an $\text{AdS}_5 \times S_5$ background. Although this duality is not yet proven, it is supported by an impressive amount of evidence. If true it is a stunning breakthrough in our thinking about string theory: it strongly supports the idea that all of the apparently non-local features of string (or string field theory) are simply due to an awkward choice of variables. In other words, there should be an alternative manifestly local formulation (in this case the SUSY Yang-Mills theory) underlying string dynamics [3].

On the other hand the duality also points to a breakthrough in our thinking about quantum field theory. The possibility of a stringy formulation of field theory could provide a powerful new way to understand hadron spectroscopy including quark confinement [4, 5]. After all, one of the most compelling mechanisms for quark confinement is the formation of color flux confining tubes between separated color sources, and a stringy reformulation of QCD might be just the long-sought change of variables that definitively clarifies the origin of these flux tubes.

Following this line of thought, Bardakci and I [6] were motivated to build a worldsheet representation of the sum of planar diagrams in a generic quantum field theory. In contrast to the AdS/CFT correspondence, for which the string side is well understood only when the field theory coupling is large, our construction bases its string description directly on the weak coupling expansion of the field theory. It thus shows that a worldsheet interpretation of field theory is generic and by no means limited to the very special circumstances of the AdS/CFT duality.

In our method we first find a worldsheet description of each planar Feynman diagram parameterized with light-cone variables. Then the stringy description we build comes out in terms of the light-cone worldsheet familiar in string theory [7]. Once each diagram is given a worldsheet description, the sum of all planar diagrams can be formulated directly on that worldsheet template, producing a string theory in which the target space variables $q(\sigma, \tau)$ interact with an Ising spin system living on the worldsheet template. It seems that these Ising spins play a role analogous to that of the AdS fifth dimension.

2 Lightcone String Basics

Let us begin by recalling the description of the lightcone worldsheet in the bosonic string theory [7]. It is most convenient to use a phase space action principle with coordinates and conjugate momenta $x^\mu(\sigma, \tau)$, $\mathcal{P}^\mu(\sigma, \tau)$.

¹Supported in part by the Department of Energy under Grant No. DE-FG02-97ER-41029.

²E-mail address: thorn@phys.ufl.edu

Then the lightcone parameterization of the worldsheet is easily specified: $x^+ = (x^0 + x^3)/\sqrt{2} = \tau$, and $\mathcal{P}^+ = (\mathcal{P}^0 + \mathcal{P}^3)/\sqrt{2} = 1$. Since \mathcal{P}^μ is the density of energy momentum on the string, the range of σ is $0 < \sigma < p^+$ where p^+ is the total + component of momentum. Then the action for the dynamics of string is simply

$$S = \int d\tau \int_0^{p^+} d\sigma \left(\dot{\mathbf{x}} \cdot \mathcal{P} - \frac{1}{2} \mathcal{P}^2 - \frac{T_0^2}{2} \mathbf{x}'^2 \right) \quad (1)$$

One gets to the more familiar configuration space action by algebraically eliminating \mathcal{P} :

$$S \rightarrow \int d\tau \int_0^{p^+} d\sigma \frac{1}{2} (\dot{\mathbf{x}}^2 - T_0^2 \mathbf{x}'^2). \quad (2)$$

For the path history version of the dynamics of string it is convenient to work with a Euclidean worldsheet, which means that we continue to imaginary τ , $i\tau \rightarrow \tau > 0$. Then the exponent in the path integrand becomes

$$iS \rightarrow - \int d\tau \int_0^{p^+} d\sigma \frac{1}{2} (\dot{\mathbf{x}}^2 + T_0^2 \mathbf{x}'^2). \quad (3)$$

Finally, it will turn out that our worldsheet description of the Feynman diagrams of field theory will require the use of dual target space variables $\mathbf{q}(\sigma, \tau)$ related to the transverse coordinates by $(\mathbf{q}', \dot{\mathbf{q}}) = (\dot{\mathbf{x}}, -\mathbf{x}'/T_0^2)$. The open string boundary conditions $\mathbf{x}' = 0$ then go to Dirichlet conditions $\dot{\mathbf{q}} = 0$, with boundary values satisfying $\mathbf{q}(p^+, \tau) - \mathbf{q}(0, \tau) = \mathbf{p}$, the total transverse momentum carried by the string. In these variables the Euclidean worldsheet action becomes

$$iS = - \int d\tau \int_0^{p^+} d\sigma \frac{1}{2} (\mathbf{q}'^2 + T_0^{-2} \dot{\mathbf{q}}^2). \quad (4)$$

We shall find a similar representation for the free field theory propagator except that the time derivative term will be absent. This is reasonable since the point particle limit of string is $T_0 \rightarrow \infty$.

3 QFT Lightcone Worldsheet

Now we turn to the worldsheet representation of the individual planar diagrams of a field theory with cubic couplings. We start with the free propagator. In the mixed $(\tau = ix^+, p^+, \mathbf{p})$ representation, the Feynman propagator for a massless scalar with $p^+ > 0$ is just $\theta(\tau) e^{-\tau \mathbf{p}^2/2p^+}/2p^+$. We choose to absorb the $1/2p^+$ factor in the *earlier* vertex to which the propagator attaches. Then in [6], we note the following remarkable identity

$$\exp \left\{ -\frac{T}{2p^+} \mathbf{p}^2 \right\} = \int_{\substack{\mathbf{q}(0)=0 \\ \mathbf{q}(p^+)=\mathbf{p}}} DcDbD\mathbf{q} e^{-S_0} \quad (5)$$

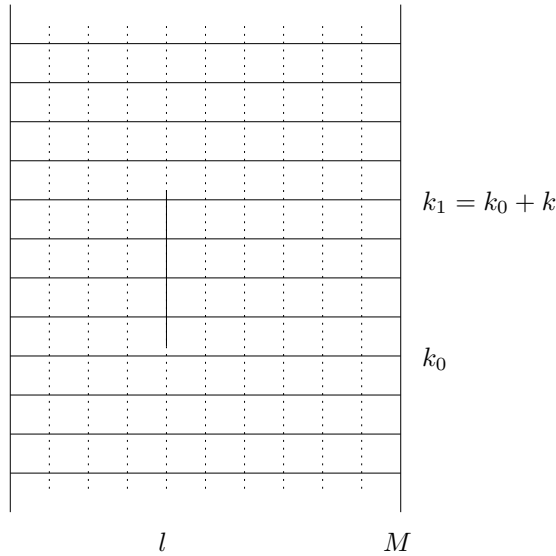
$$S_0 = \int_0^T d\tau \int_0^{p^+} d\sigma \left(\frac{1}{2} \mathbf{q}'^2 - \mathbf{b}' \cdot \mathbf{c}' \right) \quad (6)$$

where it is understood that $\mathbf{b} = \mathbf{c} = 0$ on all boundaries. This formula is central to our construction. The right side is a path integral over bosonic target space variables $\mathbf{q}(\sigma, \tau)$ together with Grassmann variables $\mathbf{b}(\sigma, \tau), \mathbf{c}(\sigma, \tau)$ defined on a rectangular worldsheet of dimensions $p^+ \times T$ just like the worldsheet of the lightcone string propagator. If $D = d + 2$ is the spacetime dimension then \mathbf{q} has d components and \mathbf{b}, \mathbf{c} each have $d/2$ components. The purpose of the Grassmann variables is to cancel the determinant factors arising from the \mathbf{q} integration. The p^+ in the denominator of the exponent on the left side appears only in the geometrical width of the worldsheet on the right side. The \mathbf{p} dependence on the left side appears only

in the Dirichlet boundary conditions on the right side. In effect the formula represents a field quantum as a composite of string bits if we discretize $p^+ = Mm = (\text{Number of bits}) \times m$. Each bit carries a single unit m of p^+ .

To give rigorous meaning to the path integral on the right side it is natural [8] to put (σ, τ) on a rectangular lattice of size $M \times N$. Here $M = p^+/m$ is the longitudinal momentum in units of m and $N = T/a$ is the evolution time in units of a . This worksheet grid is the template on which the sum over all planar diagrams is to be performed. On each site there will be target space variables $\mathbf{q}_i^j, \mathbf{b}_i^j, \mathbf{c}_i^j$, and the functional integrals over them are just ordinary integrals on the lattice.

Diagrams with n loops are represented by n line segments extended in time but at fixed σ , just as in Mandelstam's interacting string diagrams [9]. The location and length of each of these segments is summed. These line segments represent internal boundaries on which Dirichlet conditions are imposed with a different value for \mathbf{q} on each segment. The \mathbf{q} on each internal segment is independently integrated. As an example we draw the one loop self energy diagram below. The internal boundary representing the loop is indicated by the solid line segment. The dotted lines represent the *absence* of a boundary.



The loop in this figure has length ka and its earlier end is located at $\sigma = lm$ and $\tau = k_0$. The boundary value of \mathbf{q} on the internal solid line is integrated over all real values.

Consider next a general multi-loop diagram for a field theory with cubic couplings:

0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	1
1	0	0	1	0	0	0	0	1
1	0	0	1	0	0	1	0	1
1	1	0	1	0	0	1	0	0
0	1	0	1	0	0	1	0	0
0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0

We observe that the structure of the diagram is completely characterized by telling for each site whether or not it is crossed by a boundary. We can keep track of this information by introducing a two valued Ising spin s_i^j for each site with value $+1$ if the site is crossed by a boundary and with value -1 if it is not. It is also convenient to use the spin up projector $P_i^j = (1 + s_i^j)/2$ with corresponding values $1, 0$. The values $1, 0$ indicated on each temporal link $(i, j)(i, j + 1)$ of the above diagram are just the values of $P_i^j P_i^{j+1}$. The summation over all planar diagrams is just the sum over all spin configurations appropriately weighted.

The appropriate weight to use in the path integral can be read off from the usual Feynman rules. It is immediate that in the “bulk”, away from boundaries, the appropriate weight is just e^{-S_0} . On the boundaries the weighting has to enforce Dirichlet boundary conditions. For each pair of consecutive sites crossed by a boundary there must be a delta function that forces the \mathbf{q} 's on the two sites to be equal. The same effect can be obtained by adding a term $\sum_{i,j} (P_i^j P_i^{j-1} a/2m\epsilon)(\mathbf{q}_i^j - \mathbf{q}_i^{j-1})^2$ to the action with the understanding that $\epsilon \rightarrow 0$ eventually. Note that such a term brings into the action an effective $\dot{\mathbf{q}}^2$ whose coefficient $P_i^j P_i^{j-1} a^2/m^2\epsilon \sim T_{eff}^{-2}(i, j)$, where T_{eff} is an effective locally dynamical string tension. This is quite like the coupling of the target space to the AdS radial coordinate in the AdS/CFT correspondence.

4 Worksheet System for Φ^3 Field Theory

The details of the worldsheet system that sums the planar diagrams of a field theory depend on the theory. We give here the one that sums the *bare* planar diagrams of a massive scalar field theory with only cubic interactions $g\Phi^3$.

$$\begin{aligned}
T_{fi} = & \lim_{\epsilon \rightarrow 0} \sum_{s_i^j = \pm 1} \int DcDbD\mathbf{q} \exp \left\{ \ln \hat{g} \sum_{i,j} \frac{1 - s_i^j s_i^{j-1}}{2} - \frac{d}{2} \ln(1 + \rho) \sum_{i,j} P_i^j \right\} \\
& \exp \left\{ -\frac{a}{2m} \sum_{i,j} (\mathbf{q}_{i+1}^j - \mathbf{q}_i^j)^2 - \frac{a}{2m\epsilon} \sum_{i,j} P_i^j P_i^{j-1} (\mathbf{q}_i^j - \mathbf{q}_i^{j-1})^2 \right\} \\
& \exp \left\{ \frac{a}{m} \sum_{i,j} \left[A_{ij} \mathbf{b}_i^j \mathbf{c}_i^j + C_{ij} (\mathbf{b}_{i+1}^j - \mathbf{b}_i^j) (\mathbf{c}_{i+1}^j - \mathbf{c}_i^j) - B_{ij} b_i^j c_i^j - D_{ij} (b_{i+1}^j - b_i^j) (c_{i+1}^j - c_i^j) \right] \right\}
\end{aligned} \tag{7}$$

On the lattice the functional measure in this formula is rigorously defined as the product of many ordinary integration measures

$$DcDbDq \equiv \prod_{j=1}^N \prod_{i=1}^{M-1} \frac{d\mathbf{c}_i^j d\mathbf{b}_i^j}{2\pi} d\mathbf{q}_i^j \quad (8)$$

We have also employed both forms of the Ising spin variables s_i^j and $P_i^j = (1 + s_i^j)/2$. We have used the ratio of lattice constants m/a to define a dimensionless coupling constant $\hat{g} > 0$ by $\hat{g}^2 = (g^2/64\pi^3) (m/2\pi a)^{(d-4)/2}$. According to our worldsheet picture there should be a factor of \hat{g} at the beginning and end of each internal solid line. In terms of the Ising spin configuration, these points are where a spin flips. Thus the first term in the first exponent provides exactly the right factors of coupling for every planar diagram. The first term in the second exponent is just the \mathbf{q} part of S_0 . The second term in the second exponent enforces Dirichlet boundary conditions on the \mathbf{q} variables as already discussed.

The terms in the third exponent require some explanation. These terms all involve the Grassmann ghosts. Although there are many terms that we will describe in a moment, note that the Grassmann integrations on different time slices are decoupled from one another. The coefficients $(A, B, C, D)_{i,j}$ are each polynomials in the P 's associated with lattice sites at most two steps away from site (i, j) . Thus the entire expression defines a worldsheet system with completely local dynamics. These coefficients are defined in detail as follows:

$$A_{ij} = \frac{1}{\epsilon} P_i^j P_i^{j-1} + P_i^{j+1} P_i^j - P_i^{j-1} P_i^j P_i^{j+1} + (1 - P_i^j)(P_{i+1}^j + P_{i-1}^j) + \rho(1 - P_i^j) P_{i-1}^{j-1} P_{i-1}^j \quad (9)$$

$$B_{ij} = (1 - P_i^j) P_i^{j-1} P_i^{j-2} P_{i+1}^j + (1 - P_i^j) \left(P_{i+1}^j P_{i+1}^{j+1} (1 - P_{i+1}^{j-1}) + P_{i-1}^j P_{i-1}^{j+1} (1 - P_{i-1}^{j-1}) \right) \quad (10)$$

$$C_{ij} = (1 - P_i^j)(1 - P_{i+1}^j) \quad (11)$$

$$D_{ij} = (1 - P_i^j)(1 - P_{i+1}^j) P_i^{j-1} P_i^{j-2} \quad (12)$$

The $1/\epsilon$ term in A is exactly correlated with the second term in the second exponent and together they provide the properly normalized delta function in the limit $\epsilon \rightarrow 0$ that enforces Dirichlet conditions on the \mathbf{q} 's. The parameter $\rho = \mu^2 a / (md - \mu^2 a)$ appearing in the last term of the first exponent and the last term in A gives a mass μ to the scalar field. The C terms are precisely the ghost terms in S_0 that involve differences of ghost fields at adjacent sites. The spin projectors in C kill their contribution when one of the adjacent sites has spin $+1$, i.e. when it is on a boundary. The remaining ghost terms in S_0 are supplied by the fourth term in A . Without the second and third terms of A , the path integrand would be independent of the Grassmann variables on the earliest site on each internal solid line. Their presence is solely to prevent the integrations over those variables from giving zero!

Although the expressions for B and D look rather intimidating, inspection of them shows that they are designed to strategically cancel certain terms in A and C respectively. Note that they only depend on one of the components of \mathbf{b} , \mathbf{c} , written in non-bold type. Their effect is to provide the $1/p^+$ factors which we removed from the propagators and absorbed in the earlier vertex. The remarkable observation here is that these apparently nonlocal factors are supplied by a *local* modification of the ghost action. Indeed that is the profound point about the formula: the entire sum of lightcone parametrized planar diagrams is produced by a local world sheet dynamics, in spite of the prolific number of rational functions of the p^+ 's that infest the usual representation of the diagrams. I hasten to stress that the formula is strictly valid only for the bare diagrams and is complete only in space-time dimensions less than 4 for which ultraviolet divergences don't generate violations of Lorentz invariance. The full power of string theory can only be unleashed on these field theories after it is demonstrated that any counter-terms required to restore Lorentz invariance also have a local worldsheet description.

5 The Planar Yang-Mills Worldsheet

Scalar Φ^3 theory is fine as a laboratory for developing the worldsheet formalism, but we are really interested in applying the method to QCD. As 't Hooft pointed out long ago [10] the planar diagram approximation to

QCD is singled out by the $N_c \rightarrow \infty$ limit, which also suppresses internal quark loops. Thus one can focus on the worldsheet construction that sums the planar diagrams of pure Yang-Mills theory. Actually for glueball spectroscopy the large N_c limit generalizes the dominant “planar” diagrams to all those that can be drawn on a cylinder with no crossed lines. Given our lightcone methodology it is natural to work in lightcone gauge $A^+ = -A_- = 0$. Then one eliminates A_+ by solving the constraints and is left with Feynman rules for the transverse fields \mathbf{A} only. Here we restrict attention to four space-time dimensions and then it is convenient to use a complex basis $A_\wedge = (A_1 + iA_2)/\sqrt{2}$, $A_\vee = (A_1 - iA_2)/\sqrt{2}$ which is depicted by attaching an arrow to the propagator line. With this notation the only non-vanishing planar vertices are:

$$\begin{array}{c} \text{Diagram: A vertex with three lines. Line 1 (bottom-left) has an arrow pointing towards the vertex. Line 2 (bottom-right) has an arrow pointing towards the vertex. Line 3 (top) has an arrow pointing away from the vertex.} \end{array} = \frac{ga}{4m\pi^{3/2}} p_3^+ \left(\frac{p_2^\wedge}{p_2^+} - \frac{p_1^\wedge}{p_1^+} \right) \quad (13)$$

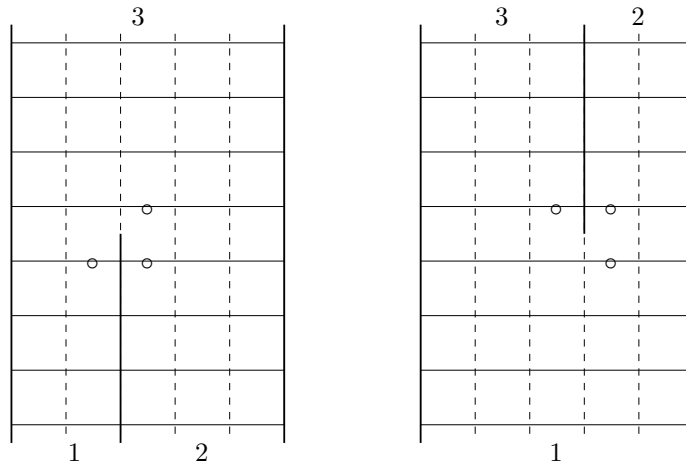
$$\begin{array}{c} \text{Diagram: A vertex with three lines. Line 1 (bottom-left) has an arrow pointing towards the vertex. Line 2 (bottom-right) has an arrow pointing towards the vertex. Line 3 (top) has an arrow pointing away from the vertex.} \end{array} = \frac{ga}{4m\pi^{3/2}} p_3^+ \left(\frac{p_2^\vee}{p_2^+} - \frac{p_1^\vee}{p_1^+} \right) \quad (14)$$

where the momenta are understood to be flowing into the vertex. The new features that must be brought into the worldsheet construction of the previous section are (1) new worldsheet degrees of freedom to account for the flow of spin (polarization) through an arbitrary planar diagram, and (2) a way to deal with the momentum dependence of the vertex. The first can be handled by introducing Neveu-Schwarz like fermionic variables, but I refer the reader to the original paper [11] for details. Here we concentrate on the issue of momentum dependence.

We must produce the momentum factors by some local feature of the worldsheet formalism. The key is to consider the expectation value of $\mathbf{q}'(\sigma, \tau)$ at some point on the worldsheet of the free propagator. Using its discretized form we find [11]

$$\frac{1}{m} \langle \mathbf{q}_l - \mathbf{q}_{l-1} \rangle = \frac{\mathbf{q}(p^+) - \mathbf{q}(0)}{p^+} = \frac{\mathbf{p}}{p^+} \quad (15)$$

which is exactly one of the terms we want to generate. Since the expectation is independent of location as long as it is an interior point on the propagator world sheet we are free to place such an insertion in the neighborhood of the end of the internal boundary that marks the spot where the gluon fission or fusion occurs. We choose candidate locations as marked by the open circles in the following worldsheet diagrams for the cubic vertex:

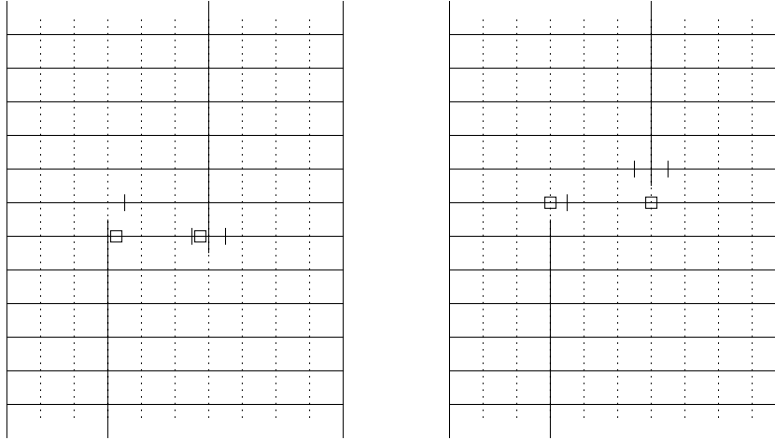


We control which combination of factors is produced by choosing the circle that lies on the appropriate gluon propagator coming into the vertex. This method locally produces everything in the vertex except the overall factor of p_3^+ . In [11] I showed how another set of Grassmann variables can be designed to locally produce these factors.

The final thing I have to say is that the worldsheet formalism described above automatically includes the quartic Yang-Mills vertex! To see how, it is enough to consider a worldsheet with two cubic vertices. Each vertex will have some combination of the \mathbf{q}' insertions just discussed. When the two insertions are at different times or on different gluon propagators nothing changes: one just has the value of the diagram with two cubic vertices. But when the two insertions are on the same time slice and on the same propagator, there is of course a fluctuation contribution to the expectation:

$$\frac{1}{m^2} \langle (q_{i+1} - q_i)(q_{j+1} - q_j) \rangle = \left(\frac{q(p^+) - q(0)}{p^+} \right)^2 + \frac{1}{a} \left[\frac{1}{m} \delta_{ij} - \frac{1}{p^+} \right] \quad (16)$$

The two situations in which this fluctuation term comes into play are illustrated in the following diagrams. The open squares show where the insertions must be to yield a fluctuation contribution. On the left is the coincidence limit of two cubics in a t (exchange) channel diagram while on the right the coincidence limit is in an s (direct) channel diagram. Remarkably the combination of the $-1/ap^+$ fluctuation terms from the two contributions exactly reproduces the Yang-Mills quartic vertex! We know that the quartic vertex is required by gauge invariance, but apparently the worldsheet formalism is clever enough to know about this subtle requirement *and* to achieve it *locally* on the worldsheet. I regard this as a dramatic indication that our way of building a worldsheet interpretation is definitely on the right track. The rest of the fluctuation term δ_{ij}/am term is not directly associated with the quartic vertex but in any case is a local $\delta(\tau)\delta(\sigma)$ worldsheet contact term, whose role is still mysterious.



6 Supersymmetry

I remark briefly that the worldsheet construction has been extended to $\mathcal{N} = 1, 2, 4$ supersymmetric Yang-Mills theories [12]. For $\mathcal{N} = 1$ SUSY one adds the appropriate fermions, and everything works as with pure Yang-Mills. For $\mathcal{N} = 2, 4$ SUSY one first adds 2 or 6 “dummy dimensions”, freezing the values of the extra components of \mathbf{q} to be zero on all worldsheet boundaries, internal as well as external. Then the extra ghost integrals exactly cancel the extra \mathbf{q} integrals. Then it turns out that the fluctuations of these dummy-dimensions generate all the required quartic interactions of extended SUSY, *exactly* as in pure Yang-Mills.

7 Conclusion: Recent and Ongoing Work

I hope I have managed here to convey the basic principles and methods underlying the worldsheet description of quantum field theory developed by Bardakci and me. We have constructed a worldsheet “template” for summing the planar diagrams in a broad range of interesting theories. There has not been time to describe several approaches we have taken toward using the formalism as a calculational tool. In [13–16] we have developed several versions of a mean field approximation to the dynamics of the worldsheet Ising spin system. Results for scalar Φ^3 theory indicate a regime at strong coupling where the mean field for the Ising spin plays a role similar to that of the AdS radial coordinate in the AdS/CFT correspondence. A similar analysis for planar QCD and SUSY gauge theories remains to be done.

The important issue of renormalization in the worldsheet formalism is wide open and under active investigation. We have emphasized that the worldsheet systems we construct reproduce all *bare* planar diagrams, which means it can be completely trusted in space-time dimensions sufficiently low to ensure the absence of ultraviolet divergences in the field theory. The worldsheet lattice we use actually cuts off all divergences, but not in a manifestly Lorentz invariant way. Consequently field theoretic ultraviolet divergences produce Lorentz violating artifacts that survive the continuum limit. Counter-terms must therefore be introduced to cancel these artifacts. Glazek has constructed a set of counter-terms for lightcone quantum field theory [17] but it is not obvious from his work that counter-terms can be chosen as *local* worldsheet modifications. Indeed, I think it is fair to say that proving that all necessary counter-terms for restoring Lorentz invariance are local on the worldsheet is tantamount to definitively establishing our worldsheet system as a bona fide representation of the fully renormalized Yang-Mills theory in four space-time dimensions.

8 Acknowledgments

I am grateful to K. Bardakci, S. Glazek, J. Greensite, S. Gudmundsson, and T. Tran for valuable discussions. This work was supported by in part by the Department of Energy under Grant No. DE-FG02-97ER-41029.

References

- [1] I. R. Klebanov, Nucl. Phys. B **496** (1997) 231 [arXiv:hep-th/9702076]; S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **499** (1997) 217 [arXiv:hep-th/9703040]; S. S. Gubser and I. R. Klebanov, Phys. Lett. B **413** (1997) 41 [arXiv:hep-th/9708005].
- [2] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231-252, hep-th/9711200.
- [3] C. B. Thorn, in *Sakharov Memorial Lectures in Physics*, Ed. L. V. Keldysh and V. Ya. Fainberg, (Nova Science, New York, 1992) 447-454, hep-th/9405069
- [4] I. R. Klebanov and M. J. Strassler, JHEP **0008** (2000) 052 [arXiv:hep-th/0007191].
- [5] J. Polchinski and M. J. Strassler, arXiv:hep-th/0003136; Phys. Rev. Lett. **88** (2002) 031601 [arXiv:hep-th/0109174]; JHEP **0305** (2003) 012 [arXiv:hep-th/0209211].
- [6] K. Bardakci and C. B. Thorn, Nucl. Phys. **B626** (2002) 287, hep-th/0110301.
- [7] P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, *Nucl. Phys.* **B56** (1973) 109.
- [8] R. Giles and C. B. Thorn, *Phys. Rev.* **D16** (1977) 366; C. B. Thorn, *Phys. Lett.* **70B** (1977) 85; *Phys. Rev.* **D17** (1978) 1073.
- [9] S. Mandelstam, *Nucl. Phys.* **B64** (1973) 205; *Phys. Lett.* **46B** (1973) 447; *Nucl. Phys.* **B69** (1974) 77; see also his lectures in *Unified String Theories*, ed. M. Green and D. Gross (World Scientific) 1986.
- [10] G. 't Hooft, *Nucl. Phys.* **B72** (1974) 461.

- [11] C. B. Thorn, Nucl. Phys. B **637** (2002) 272 [arXiv:hep-th/0203167].
- [12] S. Gudmundsson, C. B. Thorn, and T. A. Tran Nucl. Phys. B **649** (2002) 3, [arXiv:hep-th/0209102].
- [13] K. Bardakci and C. B. Thorn, Nucl. Phys. B **652** (2003) 196, [arXiv:hep-th/0206205].
- [14] K. Bardakci and C. B. Thorn, Nucl. Phys. B **661** (2003) 235, [arXiv:hep-th/0212254].
- [15] K. Bardakci, “Self consistent field method for planar ϕ^3 theory,” arXiv:hep-th/0308197.
- [16] C. B. Thorn and T. A. Tran, arXiv:hep-th/0307203.
- [17] S. D. Glazek, Phys. Rev. D **60** (1999) 105030 [arXiv:hep-th/9904029]; Phys. Rev. D **63** (2001) 116006 [arXiv:hep-th/0012012]; Phys. Rev. D **66** (2002) 016001 [arXiv:hep-th/0204171]; arXiv:hep-th/0307064.